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by
STEPHEN C. GRAVES

Technical Report No. 141

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FOR PERISHABLE INVENTORY SYSTEMS

by

STEPHEN C. GRAVES

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FOREWORD

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Co-Directors

ABSTRACT

This paper develops three distinct models for studying perishable inventory systems. The perishable items have a deterministic usable life after which they must be outdated. For each of the models, analytical expressions are found for steady-state distributions which characterize the inventory systems. Knowledge of this steady-state behavior may be used for evaluation of system performance, and for consideration of alternatives for improving system performance.

The first model considered assumes that both the demand process and the inventory replenishment process are stochastic processes that may be modelled as Poisson processes. The second and third model assume that inventory is replenished by a constant production process. The second model, assuming continuous inventory units, has Poisson demand requests with the size of each request distributed as an exponential random variable. The third model has Poisson demand requests with all demands being for a single unit.

1. INTRODUCTION

This paper is concerned with the development of models for studying inventory systems in which the item is perishable. In particular, the item is assumed to have a lifetime of m periods, after which it is said to be outdated and cannot be issued to service demand. Examples of such items might be blood in a blood-banking system, packaged food items or chemical supplies. This paper considers three distinct models of perishable inventory systems. For each of these models, a steady-state distribution of the inventory status is found. Knowledge of this steady-state behavior may be used for evaluating a system's performance, and for considering alternative means for improving the system.

Much of the literature studying perishable inventory systems has dealt with finding optimal periodic ordering policies. Typical of this work is the work by Nahmias [10], [11], [12], Fries [4], and Cohen [5]. Implicit in this work is the assumption that an order can always be filled by the supplier. The analysis necessary for finding optimal policies requires an m-dimension state space where m periods is the lifetime of the item. Consequently this work has dealt with determining and evaluating approximate policies. The most common policy considered is an order-up-to policy: each period an order is placed so to bring the total inventory up to a preset level. Related to this work is that of Jennings [7] who used a simulation model to study a blood banking system. With the simulation, he was able to evaluate various ordering policies, and was able to consider the effects of centralized control on a regional multi-hospital blood banking system.

The models developed in this paper are distinct from the work on periodic ordering policies, in that here specific replenishment policies are assumed. The replenishment or ordering policy is not a decision

variable; rather, given a replenishment policy, the models are descriptive of the stochastic system behavior. In this sense, this paper is related to the descriptive Markov models for perishable inventory systems presented by Pegels and Jelmert [13], Brodheim, Derman and Prastacos [1], and Chazan and Gal [2].

Pegels and Jelmert [13] model a blood-bank inventory by a Markov chain. This work has been criticized by Kolesar [8]. Brodheim, et al. study an inventory system assuming a constant replenishment policy; that is, each period n units of new inventory are supplied. They model the system as a Markov chain in discrete time, but are unable to solve the steady-state equations. They do establish upper and lower bounds on various measures of interest, such as expected outdates and expected shortages, provided that n, the supply quantity, is less than the expected demand rate. Chazan and Gal consider an inventory system operating under an order-up-to replenishment policy. They use a Markov model to find bounds on the expected outdates assuming a general demand process. For Poisson demand, they are able to find the steady-state behavior for the system for the Markov chain in continuous time. That is, whenever a unit is requested or a unit becomes outdated, it is assumed that a new unit is instantly acquired to replace the old unit to maintain a constant inventory level; note that for such a system there can never be any shortages. Using the results from the continuous case, tight bounds are developed for the Markov chain in discrete time.

The remainder of the paper is organized as follows: the next section considers an inventory system for which the replenishment process is stochastic and may be modeled as a Poisson process. Sections 3 and 4 consider models in which there is constant replenishment. The problem in these sections is similar to that considered in [1]; however, here we are able to present analytical results. Section 3 considers a continuous

model in which the size of demand requests is exponential. Section 4 assumes that demand requests are all for single units. In all models it is assumed that requests arrive as a Poisson process. The final section gives a summary and discussion of the results of the paper.

2. STOCHASTIC REPLENISHMENT

Consider the inventory system assuming that both the demand process and the replenishment process is Poisson. Demand requests occur at rate μ, and a request is always for one unit. The inventory is replenished at rate λ with unit replenishments. This type of replenishment process may be appropriate whenever there is uncertainty in the supply procedure. For instance, in the case of blood inventory, to a certain extent supplies depend on volunteer donors who may be thought of as behaving as a random process. Alternatively, in a production environment, the production process may be very sensitive to human or random elements, and consequently the time to produce a successful unit is variable. An example of this might be a repair depot where the time to diagnose the failure or problem is random, and consumes most of the repair time. In addition to stochastic demand and replenishment, assume that the item has a lifetime of m periods, the issuing policy is to issue the oldest usable item, and that demand requests that cannot be filled from inventory are turned away or handled exogenous to the system.

This system can be modeled as a Markov chain in continuous time. One characterization of the state space is by a complete account of the inventory. That is, the state variable would be $\{N(t), A_1(t), \ldots A_{N(t)}(t)\}$ where $N(t)$ is the number of units in inventory at time t, and $A_i(t)$ is the age of the i^{th} unit in inventory where the units are ranked by age, for $i=1,2,\ldots,N(t)$. In this form, however, the analysis of the process is difficult, if not impossible, due to the varying dimensions of the state space.

As an alternative, consider the process $A(t)$ where $A(t)$ is the age of the oldest unit in inventory. Given the Poisson assumptions for demand

and supply, and assuming we have <u>no other information</u> about the inventory status, then this process is Markov and will provide a sufficient characterization of the system for our purposes. The process $A(t)$ can take on any value α for $0 \leq \alpha < m$. The state space for $A(t)$ must also include the state when there is no inventory in the system; this will be denoted by $A(t) = E$. Note that $A(t) = E$ is distinct from $A(t) = 0$.

The transitions for $A(t)$ can be characterized by examining the interval $(t, t+\Delta)$ for the following three cases, where $\Delta > 0$ is small.

<u>Case 1</u> - Suppose that $A(t) = \alpha$ for $0 \leq \alpha < m-\Delta$. Then at $t+\Delta$, we have that either a demand request occurred (with probability $\mu\Delta$) or it did not [with probability $(1-\mu\Delta)$]. If no demand occurs, then $A(t+\Delta) = \alpha+\Delta$. If a demand occurs, then the oldest unit (age $= \alpha$) is issued, and $A(t)$ becomes the age of the next oldest unit in inventory or $A(t) = E$ if the issued unit was the only unit in inventory. Define τ to be a random variable for inter-arrival time of units into inventory; τ is an exponential random variable with mean $1/\lambda$. If $\tau < \alpha+\Delta$, then there is a second unit in inventory and $A(t+\Delta) = \alpha+\Delta-\tau$. If $\tau \geq \alpha+\Delta$, this implies that the oldest unit was the only unit in inventory, and the system is now empty, $A(t+\Delta) = E$. Note that for this case a replenishment epoch has no effect on the process.

<u>Case 2</u> - Suppose that $A(t) = \alpha$ for $m-\Delta \leq \alpha<m$. Then at $t+\Delta$ the oldest unit will have been issued or outdated. Thus, using similar reasoning as in Case 1, $A(t+\Delta) = m-\tau$ for $\tau \leq m$, $A(t+\Delta) = E$ for $\tau > m$.

<u>Case 3</u> - Suppose that $A(t) = E$. In $(t, t+\Delta)$, the system will either remain empty [with probability $(1-\lambda\Delta)$, $A(t+\Delta)=E$] or a new

replenishment will arrive [with probability $\lambda\Delta$, $A(t+\Delta)=0$]. Given the exponential interarrival times for replenishments, the probability of a replenishment in $(t, t+\Delta)$ is independent of the time of the last previous replenishment. Note also that this case is unaffected by demand requests, since they are not backordered.*

Considering the above three cases, the process $\{A(t)\}$ is clearly Markov since the transition laws for the interval $(t, t+\Delta)$ depend only on the state $A(t)$ at time t.

Define $p(x,t)$ to be the probability density for $A(t) = x$, $0 \leq x < m$, and let $\pi(t)$ be the probability that $A(t) = E$. The equations of motion can be shown to be

$$(1) \qquad \frac{\partial p(x,t)}{\partial x} + \frac{\partial p(x,t)}{\partial t}$$

$$= -\mu p(x,t) + \mu \int_{y=x}^{m} p(y,t)\lambda e^{-\lambda(y-x)}dy + p(m,t)\lambda e^{-\lambda(m-x)},$$

$$0 \leq x < m,$$

and

$$(2) \qquad \frac{d}{dt}\pi(t) = -\lambda\pi(t) + \mu \int_{y=0}^{m} p(y,t)e^{-\lambda y}dy + p(m,t)e^{-\lambda m}.$$

In addition to this, the total probability mass must be 1:

$$(3) \qquad \pi(t) + \int_{x=0}^{m} p(x,t)dx = 1.$$

By differentiating both sides of (3) with respect to t, and substituting (1) and (2), we find a boundary condition for the process:

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* The current model is easily extended to allow backorders provided that $\lambda > \mu$.

(4) $p(0,t) = \lambda\pi(t)$.

The steady-state equations are found by letting $t \to \infty$, $p(x,t) \to p(x)$, and $\pi(t) \to \pi$. Equations (1), (3), and (4) are now rewritten as

(5) $$\frac{d}{dx} p(x) = -\mu p(x) + \mu \int_{y=x}^{m} p(y)\lambda e^{-\lambda(y-x)} dy + p(m)\lambda e^{-\lambda(m-x)},$$

$$0 \leq x < m,$$

(6) $$\pi + \int_{x=0}^{m} p(x)dx = 1,$$

and

(7) $$p(0) = \lambda\pi.$$

The solution to (5), (6), and (7) is found to be

(8) $$p(x) = Ke^{(\lambda-\mu)x} \qquad \text{for } 0 \leq x < m$$

(9) $$\pi = K/\lambda$$

(10) $$K = \lambda(\lambda-\mu)/[\lambda e^{(\lambda-\mu)m} - \mu]$$

provided that $\lambda \neq \mu$. For the special case when $\lambda = \mu$, the solution is (8), (9), and

(11) $$K = \lambda/(m\lambda + 1).$$

From (8) - (10) we can now express analytically four measures of system performance where an outdate occurs when a unit reaches age m and cannot be issued, and a shortage occurs when a demand request is turned away because there is no inventory.

a) the expected number of outdates per time unit ($\bar{O}$).

b) the expected number of shortages per time unit ($\bar{S}$).

c) the expected age of an issued unit ($\bar{A}$).

d) the expected number of units in inventory ($\bar{I}$).

Using simple probabilistic arguments, we have

$$(12) \qquad \bar{O} = p(m)$$

$$(13) \qquad \bar{S} = \mu\pi$$

$$(14) \qquad \bar{A} = \int_{x=0}^{m} xp(x)dx \Big/ \int_{x=0}^{m} p(x)dx$$

$$(15) \qquad \bar{I} = \int_{x=0}^{m} p(x)(1+\lambda x)dx$$

The expressions (12) - (15) can be used to evaluate a system, given the system parameters λ, μ, and m. These expressions can also be used to examine the sensitivity of the measures to the system parameters. For instance, taking $\lambda = 1.0$, $\mu = 1.0$, $m = 20$, as a base case Figures 1, 2, and 3 show the effect of varying the replenishment rate, the demand rate, and the unit lifetime, respectively.

The present model assumes that demand requests arrive as a Poisson process, and are for single units. The model can be easily extended to the case where the size of the demand request is distributed as a geometric random variable d. That is, for d being the number of units demanded,

$$(16) \qquad pr(d=n) = (1-p)^{n-1}p \qquad \text{for } n=1,2,\ldots$$

$$(17) \qquad E(d) = 1/p$$

The analysis of this problem is similar to the previous case except for differences in the transition laws when demand requests occur. Define τ_s to be the sum of d replenishment interarrival times, where each interarrival time is an independent exponential random variable with mean $1/\lambda$.

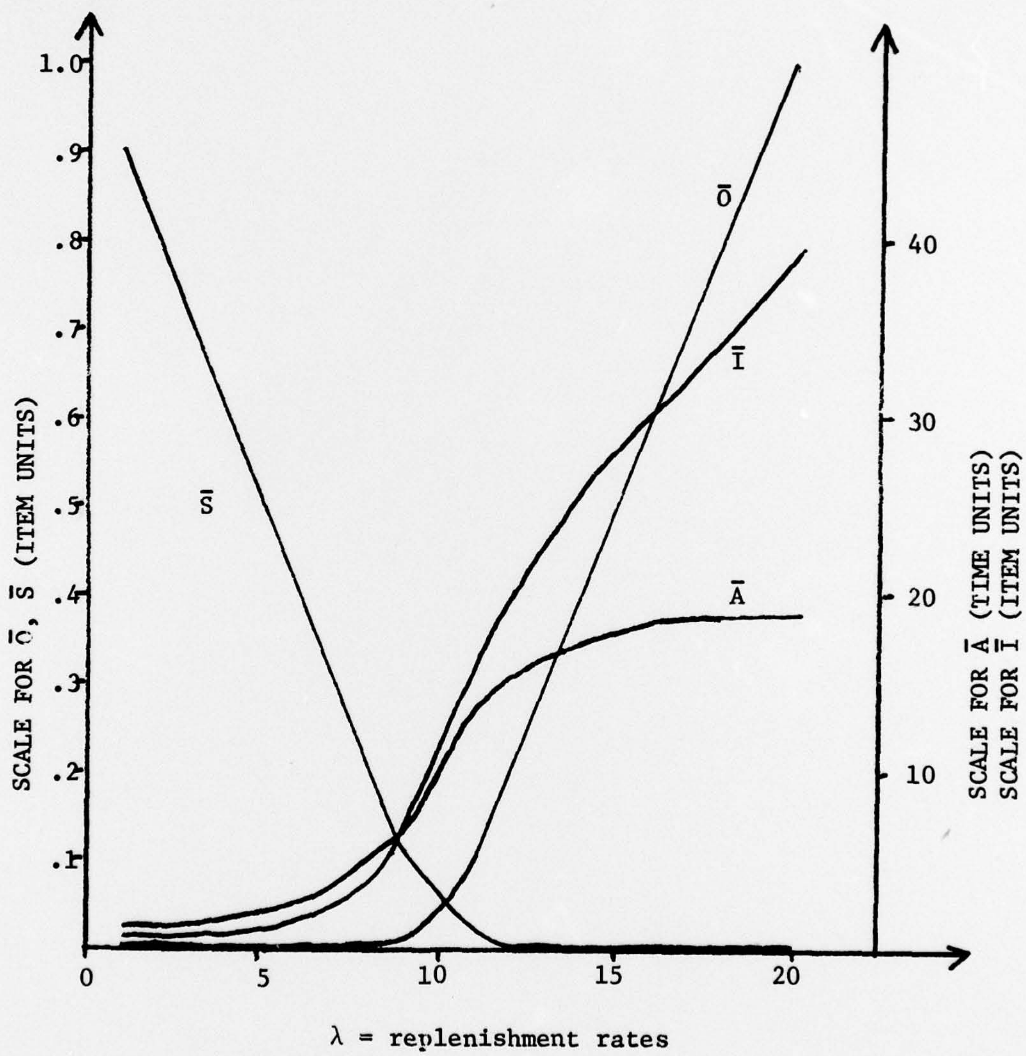

FIGURE 1: $.1 \leq \lambda \leq 2$, $\mu = 1$, $m = 20$

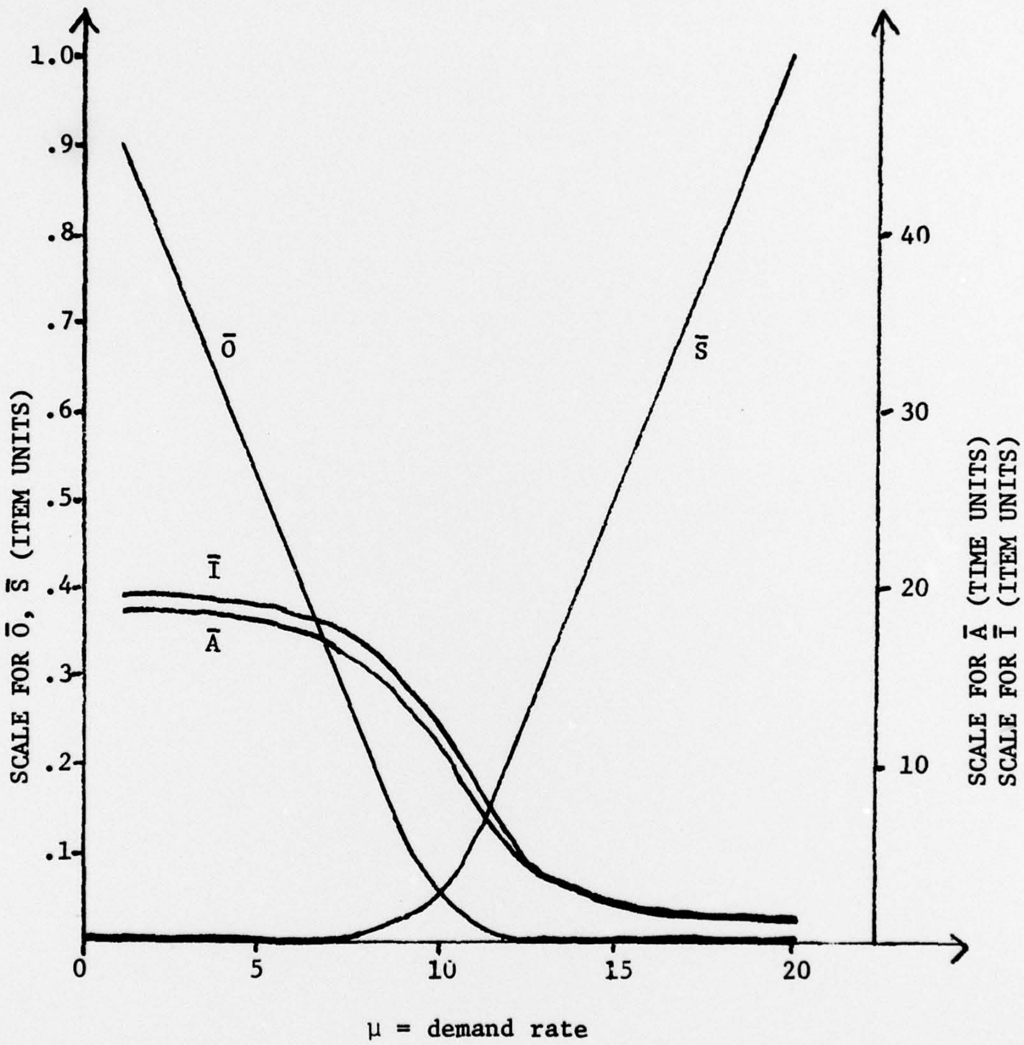

FIGURE 2: $\lambda = 1$, $.1 \leq \mu \leq 2$, $m = 20$

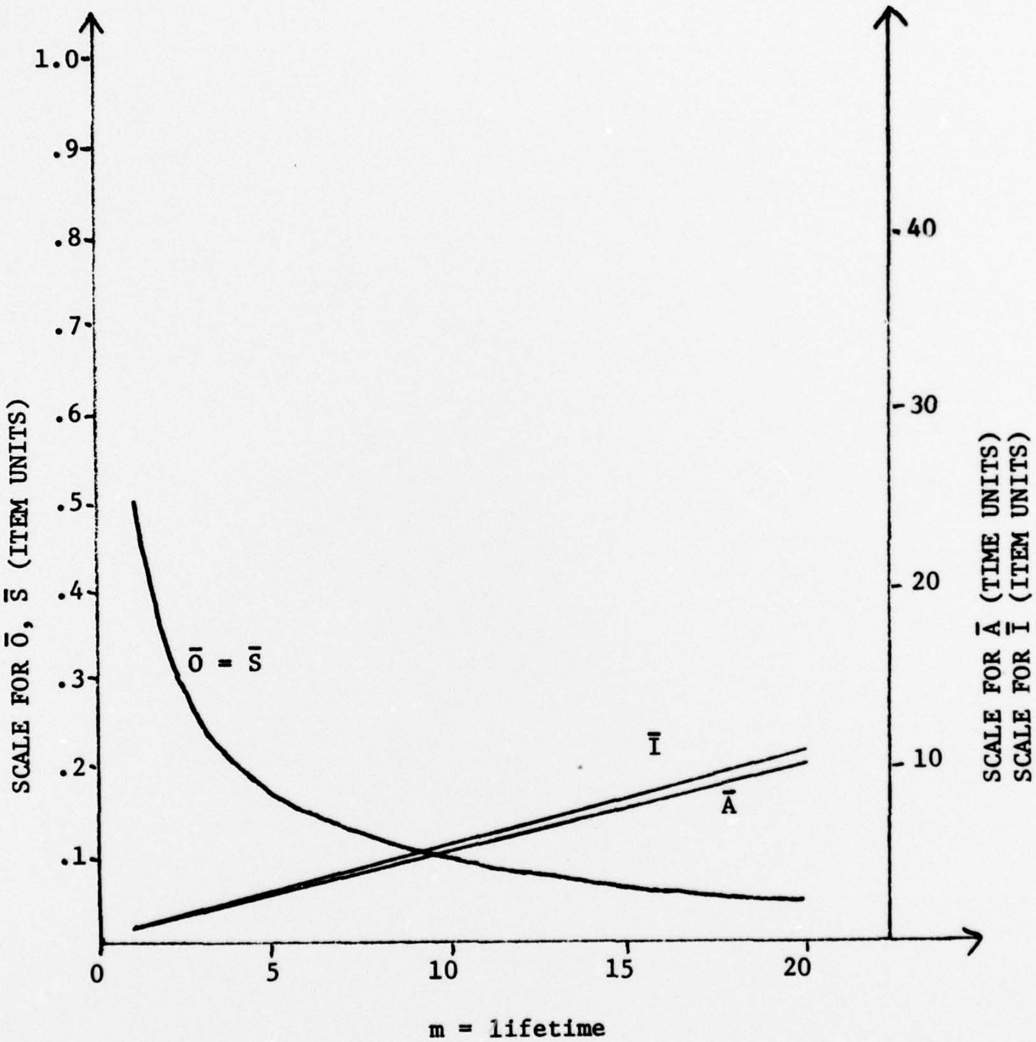

FIGURE 3: $\lambda = 1$, $\mu = 1$, $1 \leq m \leq 20$

Since d is a geometric random variable, τ_s is a compound geometric distribution of exponential random variables. It can be shown that τ_s is an exponential random variable with mean $1/p\lambda$. Now for $A(t) = \alpha < m - \Delta$, if a demand request occurs during $(t, t+\Delta)$, then $A(t+\Delta) = \alpha + \Delta - \tau_s$ if $\tau_s < \alpha + \Delta$, or $A(t+\Delta) = E$ if $\tau_s \geq \alpha + \Delta$. Note that τ_s corresponds to the removal of d units from inventory. However, if $m-\Delta \leq A(t) < m$, then the oldest unit is outdated during $(t, t+\Delta)$ with probability $1 - \mu\Delta$ (i.e. no demand). In this instance, letting τ be a single replenishment interarrival time, we have $A(t+\Delta) = m-\tau$ for $\tau \leq m$ and $A(t+\Delta) = E$ for $\tau > m$.

Using the same reasoning as in the previous case, the steady-state equations for this system are

$$(18) \qquad \frac{dp(x)}{dx} = -\mu p(x) + \mu \int_{y=x}^{m} p(y)\, \hat{\lambda} e^{-\hat{\lambda}(y-x)} dy + p(m)\, \lambda e^{-\lambda(m-x)}, \qquad 0 \leq x < m,$$

and equations (6) and (7), where $\hat{\lambda} = p\lambda$. The solution to the three equations (6), (7), and (18) is

$$(19) \qquad p(x) = K \left[e^{(\hat{\lambda}-\mu)x} + \left(\frac{\lambda-\hat{\lambda}}{\mu}\right) e^{\lambda x} e^{-m(\lambda+\mu-\hat{\lambda})} \right]$$

$$(20) \qquad \pi = \frac{K}{\lambda} \left[1 + \left(\frac{\lambda-\hat{\lambda}}{\mu}\right) e^{-m(\lambda+\mu-\hat{\lambda})} \right]$$

$$(21) \qquad K = \left[\frac{\hat{\lambda}-\mu-\lambda}{\lambda(\hat{\lambda}-\mu)} + e^{(\hat{\lambda}-\mu)m} \left(\frac{\lambda-\hat{\lambda}}{\lambda\mu} + \frac{1}{\hat{\lambda}-\mu} \right) \right]^{-1}$$

Equations (12), (14), and (15) may be used to determine expected outdates, expected age*, and expected inventory level. The expected shortages per time unit, however, now consist of shortages which occur when there is no inventory, and shortages which occur when a particular

* Here, the expected age needs to be reinterpreted to be the expected age of the oldest unit issued to fill a demand request.

demand request depletes the inventory present. The component of shortages which occur when the system is empty is $\mu\pi/p$, since the expected size of a request if $1/p$ and the expected number of requests per unit time when the inventory is zero is $\mu\pi$. To determine the expected shortages caused by demand requests in excess of present (non-zero) inventory, suppose $A(t) = x$ and a demand request occurs. Letting I be the total inventory level at time t, then given $A(t) = x$ we have

$$\text{(22)} \qquad pr[I=n|A(t)=x] = \frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} \qquad \text{for } n=1,2,...$$

Combining (22) with the distribution for the size of a demand request (16), the expected shortages conditioned on $A(t) = x$ and the occurance of a demand request, denoted as $\bar{S}(x)$, is

$$\text{(23)} \qquad \bar{S}(x) = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} (m-n) \cdot pr[I=n|A(t)=x] \cdot pr(d=m)$$

$$= \frac{(1-p)e^{-\lambda xp}}{p} \approx \frac{(1-p)e^{-\hat{\lambda}x}}{p}$$

Hence, for this system, the expected shortages per time unit may be written as

$$\text{(24)} \qquad \bar{S} = \mu\pi/p + \mu \int_0^m p(x)\bar{S}(x)dx$$

$$= \mu\pi/p + K[1 - e^{-m(\lambda+\mu-\hat{\lambda})}](1-p)/p.$$

where the first component is shortages when the system is empty, and the second component is shortages from partially-filled orders.

3. CONSTANT REPLENISHMENT WITH EXPONENTIAL DEMAND REQUESTS

Consider an inventory system for which the replenishment process is constant; that is, inventory is continuously produced at a constant rate of c units/time unit. This type of replenishment is found in any continuous production facility, such as a chemical plant or a refinery, where it is very costly to shutdown or interrupt the process. For this system, it is assumed that demand requests arrive as a Poisson process at rate μ with the request size distributed as an exponential random variable with mean $1/\gamma$. Implicit in this model is the assumption that inventory may be treated as a continuous entity. In addition, it is again assumed that inventory is issued oldest first, inventory expires at age m, and inventory shortages are not backordered.* The time unit for the system is defined such that the production rate $c = 1$.

Similar to the previous system, this inventory system may be characterized as a Markov process with state variable $A(t)$ corresponding to the age of the oldest unit in inventory. Note, however, that $A(t)$ now also represents the amount of inventory present. The process $\{A(t)\}$ ranges over the interval $[o,m]$. The transitions for the process are described by considering the following two cases:

Case 1 - Suppose $A(t) = \alpha$, $0 \leq \alpha < m-\Delta$. Then $A(t+\Delta) = \alpha+\Delta$ if no demand requests occur in $(t, t+\Delta)$. If a demand occurs, then $A(t+\Delta) = (\alpha+\Delta-\tau)^+$, where τ is the exponential random variable for the size of the demand request.

Case 2 - Suppose $A(t) = \alpha$, $m-\Delta \leq \alpha \leq m$. At $t+\Delta$, if a demand has occured,

* The backorder case is a simple extension of the current model provided that $\mu < \gamma$.

then $A(t+\Delta) = (m-\tau)^+$. Otherwise, $A(t+\Delta) = m$. Note that in the latter case, inventory is being continuously outdated.

Define $p(x,t)$ to be the probability density for $A(t) = x$, $0 \leq x \leq m$, and $\pi(t)$ to be the probability mass that $A(t) = m$. The equations of motion for this system are

(25a) $$\frac{\partial}{\partial x} p(x,t) + \frac{\partial}{\partial t} p(x,t)$$

$$= -\mu p(x,t) + \mu\pi(t)[\gamma e^{-\gamma(m-x)}] + \int_{y=x}^{m} \mu p(y,t)\gamma e^{-\gamma(y-x)} dy,$$

for $0 \leq x \leq m$,

and

(25b) $$\frac{d}{dt}\pi(t) = -\mu\pi(t) + p(m,t).$$

In addition, we have the following boundary conditions:

(26) $$p(0,t) = \int_{y=0}^{m} \mu p(y,t)e^{-\gamma y} dy + \mu\pi(t)e^{-\gamma m}.$$

By letting $t \to \infty$, $p(x,t) \to p(x)$, and $\pi(t) \to \pi$, the steady-state equilibrium equations are found to be

(27) $$\frac{d}{dt} p(x) = -\mu p(x) + \mu\pi\gamma e^{-\gamma(m-x)} + \int_{y=x}^{m} \mu p(y)\gamma e^{-\gamma(y-x)} dy,$$

for $0 \leq x \leq m$,

(28) $$0 = -\mu\pi + p(m), \text{ and}$$

(29) $$p(0) = \int_{y=0}^{m} \mu p(y)e^{-\gamma y} dy + \mu\pi e^{-\gamma m}.$$

The solution to (27) - (29), using the fact that total probability mass is one, is

(30) $p(x) = Ke^{-(\mu-\gamma)x}$, $0 \leq x \leq m$,

(31) $\pi = \frac{1}{\mu} p(m)$, where

(32) $K = \dfrac{\mu(\mu-\gamma)}{\mu - \gamma e^{-m(\mu-\gamma)}}$

provided $\mu \neq \gamma$. For $\mu = \gamma$, the solution is (30), (31), and

(33) $K = \mu/(1+m\mu)$.

By probabilistic reasoning, we can state analytic expressions for the expected outdates per time unit ($\bar{O}$), the expected shortages per time unit ($\bar{S}$), the expected age of the oldest unit supplied for a demand request ($\bar{A}$), and the expected inventory level ($\bar{I}$):

(34) $\bar{O} = \pi$

(35) $\bar{S} = (\frac{1}{\gamma})\ p(0)$

(36) $\bar{A} = \int_{x=0}^{m} xp(x)dx + m\pi$

(37) $\bar{I} = \int_{x=0}^{m} xp(x)dx + m\pi$

Inventory units are being continuously outdated at the unit rate of production (c=1) whenever $A(t) = m$ which occurs with probability mass π. To find the expected shortages, if a demand request is larger than available inventory, then due to exponential demand sizes the expected number of unfilled requests is $1/\gamma$; $p(0)$ reflects the frequency of occurance of demand requests which completely deplete the inventory. Note also that the expected age of the oldest unit issued equals the expected inventory level; this is true since the age of the oldest unit equals the inventory level, and since demands occur at random epochs. Given the expressions (34) - (37) as functions of the system parameters μ, γ, and m, the model may be easily used for system evaluation and trade-off studies.

4. CONSTANT REPLENISHMENT WITH UNIT DEMAND REQUESTS

The system studied in this section is identical to that studied in the previous section except that all demand requests are for one unit of inventory. Again the system may be characterized by the process $\{A(t)\}$ for $A(t)$ being the age of the oldest unit in inventory at time t. To analyze this system, we will not study the equilibrium flow equations for the system, but rather will draw upon results from queueing theory. It is useful to think of the system as a single-server queue, where the production process is the server, the inventory is a positive backlog in the queue, and a demand request is a customer arriving at the queue requiring one unit of service. In this framework, the system may be modeled as a capacitated $M|D|1$ queue, where the process $\{m-A(t)\}$ corresponds to the virtual waiting time for the $M|D|1$ queue. When $A(t) = m$, the server is idle and the queue empty; for $A(t) = \alpha < m$, the server is busy with total queue length Q equal to the smallest integer greater than $m-A(t)$, and with a virtual waiting time of $m-A(t)$. Since $A(t)$ is defined to be non-negative, this implies that the queueing system has capacity m; queue arrivals which would put the virtual waiting time in excess of m are truncated so that the virtual waiting time equals m.

Hence, the determination of the distribution for $\{A(t)\}$ is equivalent to finding the distribution of the virtual waiting time for a bounded $M|D|1$ system. Unfortunately, to this author's knowledge, this problem is unsolved (see Gavish and Schweitzer [5] who considered a closely-related problem). However, results exist for finding the distribution for a discrete approximation of $\{A(t)\}$. The approximation is analogous to using queue length as a surrogate for the virtual waiting time.* Here, $A(t) = m$ corresponds to an empty queue, $m-1 \leq A(t) < m$ corresponds to a queue length

* This is appropriate here since each customer requires one unit of service.

of one, and so on. Defining p_n to be the probability that $n \leq A(t) < n+1$ for $n=0,1,\ldots,m-1$ and p_m to be the probability that $A(t) = m$, then the results of Keilson [8] for the finite-capacity M|G|1 system can be extended to the current problem for $\mu < 1$:

(38) $$p_o = 1 - (1-p_m)/\mu$$

(39) $$p_i = Ke_{m-i} \qquad i=1,2,\ldots,m$$

(40) $$K = [e_o + \mu(\sum_{i=0}^{m} e_i)]$$

where e_i is the steady-state probability of queue length i for the M|D|1 queue with infinite capacity. That is (see [6])

(41) $$e_o = 1 - \mu$$

(42) $$e_1 = (1-\mu)(e^{\mu}-1)$$

(43) $$e_i = (1-\mu)\left[\sum_{k=0}^{i-1} \frac{(-1)^k \mu^k e^{\mu(i-k)} (i-k)^k}{k!} - \sum_{k=0}^{i-2} \frac{(-1)^k \mu^k e^{\mu(i-k-1)} (i-k-1)^k}{k!}\right]$$

for $i=2,3,\ldots$

Keilson [8] also presents a solution for $\mu > 1$; however this result is in terms of a Green's function and will not be discussed here.

Now (38) - (40) can be used to find the four evaluative measures of system performance:

(44) $$\bar{O} = p_m$$

(45) $$\bar{S} = \mu p_o$$

(46) $$\bar{A} = \sum_{i=0}^{m} ip_i$$

(47) $$\bar{I} = \sum_{i=0}^{m} ip_i$$

These expressions are analogous in their derivation to (34) - (37). However, the above measures are approximate since they correspond to a discrete representation of the process for the oldest unit in inventory.

The system considered in this section is identical to that considered by Brodheim, et al. [1]; however the results are quite different. In [1] the system is modeled as a Markov chain in discrete time on discrete state space; given this representation, the paper establishes lower and upper bounds on various measures of interest provided the demand rate (μ) exceeds the production rate ($c=1$). The model presented in this section models the system in continuous time on a continuous state space; using results from queueing theory, analytical results are found for the evaluative measures provided the demand rate is less than the production rate ($\mu < 1$).

5. DISCUSSION

This paper has considered three distinct models for studying inventory systems for perishable items. The key similarity across the three models is that for each model assumptions were made so that the system could be characterized by a one-dimension stochastic process. For each of the models, analytical expressions have been found for four performance measures: expected outdates, expected shortages, average age of an issued unit, and expected inventory level. It is felt that these expressions can be used for system evaluation and design decisions for systems for which the assumptions of the respective model are reasonable.

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